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On the rational solutions of the $\widehat{su}(2)_k$ Knizhnik-Zamolodchikov equation

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Abstract. We present some new results on the rational solutions of the Knizhnik-Zamolodchikov (KZ) equation for the four-point conformal blocks of isospin I primary fields in the $SU(2)_k$ Wess-Zumino-Novikov-Witten (WZNW) model. The rational solutions corresponding to integrable representations of the affine algebra $\widehat{su}(2)_k$ have been classified in [1,2]; provided that the conformal dimension is an integer, they are in one-to-one correspondence with the local extensions of the chiral algebra. Here we give another description of these solutions as specific braid-invariant combinations of the so called regular basis introduced in [3] and display a new series of rational solutions for isospins $I = k+1\,,\ k \in \mathbb{N}$ corresponding to non-integrable representations of $\widehat{su}(2)_k$.

PACS. 11.25.Hf Conformal field theory, algebraic structures – 02.20.Uw Quantum groups – 11.30.Ly Other internal and higher symmetries

1 Introduction

The $2D$ nonlinear σ-model with a suitably normalized WZ term, known as WZNW model [4], is a conformally invariant (and therefore integrable) field theory with a huge internal symmetry, beautiful geometric structure at the classical level and rich algebraic content in the quantized case. The model describes a closed (respectively, open, for the so called boundary model) string moving freely on a Lie group manifold G. After choosing the group, the only parameter left which fixes the theory is a positive integer k playing the role of WZ term coupling constant. Here we will only consider the case of the compact group $G = SU(2)$.

To solve the model, one can use different approaches in both the classical and the quantum cases. In the *axiomatic* approach to the quantized model one constructs the space of states as a direct sum of superselection sectors (tensor products of integrable representations of the corresponding left and right current algebras, both of which appear to be affine algebras of the type $\hat{\mathcal{G}}_k$ where $\mathcal{G}$ is the Lie algebra of G and k is the level). Each sector is generated from the vacuum by a primary field. The interplay of affine and conformal invariance leads to linear systems of partial differential equations for the correlation functions of primary fields (conformal blocks), one for each set of chiral variables. These are the famous KZ equations [5,6] determining, in principle, the chiral structure of the theory; the correlation functions of the $2D$ theory are recovered by combining the left and the right conformal blocks (which are, typically, multivalued) in such a way that $2D$ locality is restored [7]. In some cases this can be done in different ways constrained by modular invariance of the WZNW partition function; for $G = SU(2)$ this leads to the ADE classification of [8].

The *canonical* quantization of the chirally split WZNW model [9–15] leads to a description in terms of chiral fields revealing the quantum group (QG) invariance of their exchange algebras which is the quantum counterpart of the Poisson-Lie invariance of the underlying classical theory (see [16] for a recent comprehensive exposition of the classical situation and [17], focused on the boundary WZNW model). The fact that the monodromies of the chiral correlation functions are related to $U_q(\mathcal{G})$ $6j$-symbols (for q an even root of unity, $q = e^{\pm i\frac{\pi}{k+2}}$ for $G = SU(2)$) has been known for a long time [18]. On the other hand, it is clear that the true "internal" symmetry of the model is much more involved (see *e.g.* [19] and references therein for a recent analysis of the relation between weak C^*-Hopf algebras and rational conformal field theories). A plausible way out of this apparent contradiction would be the assumption [20] that $U_q(\mathcal{G})$ plays the role of a generalized "gauge" group on the *extended* (chirally split) WZNW so that one is facing an alternative analogous to choosing unitary or covariant gauges in gauge theories, the latter necessarily including unphysical states. In the case at hand

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this means that we have to consider indecomposable representations of both the conformal current algebra and of the quantum group.

For the $SU(2)_k$ WZNW model the minimal extension should involve primary fields with isospins covering at least twice the range of the unitarizable representations, $0 \leq 2I \leq 2k+2$, since the indecomposable QG counterparts relate I with $k+1-I$. Allowing for non-integrable I, one has to expect the extended theory to be logarithmic [21,22] (see *e.g.* [23] and references therein). A fact, related to the latter, is that the commonly used bases of KZ solutions [24,25] are ill defined because some mutual normalizations become inevitably infinite; fortunately, (regular) bases are known [3] that remain meaningful in this wider range of the isospins [26]. The elements of all these bases are given, up to a prefactor which is an algebraic function of coordinate differences, by multi-contour integrals in the complex domain (identified, for n-point correlation functions, with the Riemann sphere $\mathbb{C}P^1$ with $n-1$ punctures). The braiding properties of the elements of the regular bases have been also displayed in [3], the corresponding elementary braid matrices being triangular and well defined.

Let us consider the set of four-point conformal blocks of $SU(2)_k$ WZNW chiral fields of isospin I. Among them there is a distinguished set given by *rational* functions. For $0 \leq 2I \leq k$ the importance of these has been elucidated in [1,2,27] (see also [28] where the more general problem of finding all *algebraic* KZ solutions has been solved) – they are in one-to-one correspondence, provided that the conformal dimension $\Delta_I = \frac{I(I+1)}{k+2}$ is an integer, with the possible extensions of the corresponding chiral algebra (the algebra of observables, in this case the current algebra). In fact, only primary fields with integer or half-integer conformal dimensions which are also local with respect to themselves can have rational four-point functions. Local commutativity (in the chiral sense) singles out the D_{even} series and the exceptional E_6 and E_8 models with diagonal pairing in the ADE classification.

The main objective of the present paper is to extend the results of [1,2] finding rational solutions for non-integrable values of $I\,(>\frac{k}{2})$ as well. After introducing our basic conventions and notations in Section 2, in Section 3 we employ an alternative description of the rational solutions of the KZ equation as braid invariant linear combinations of the regular basis vectors. This leads to nontrivial relations even in the known cases. New rational solutions for $I=k+1$ are displayed in the last Section 4 where we also analyze their properties. We hope to be able to present an exhaustive study of this subject in the near future [29].

2 KZ equation and braiding properties of the regular basis

We will give here a short list of all needed notions and formulas; for the lack of space we refer for details to [1–3] and [26].

The conformal block containing four primary fields $\Phi_I(z)$ of isospin I can be expressed as

$$\langle\, \Phi_I(z_1)\, \Phi_I(z_2)\, \Phi_I(z_3)\, \Phi_I(z_4)\,\rangle \;=\; D_I\left(\underline{\zeta}, \underline{z}\right)\, f_I(\xi,\eta)\,,$$

$$D_I(\underline{\zeta},\underline{z}) = (\xi_1+\xi_2)^{2I}\left(\frac{\eta_1+\eta_2}{\eta_1\eta_2}\right)^{2\Delta_I} \equiv (\zeta_{13}\zeta_{24})^{2I}\left(\frac{z_{13}z_{24}}{z_{12}z_{34}z_{14}z_{23}}\right)^{2\Delta_I} \qquad (1)$$

(see [2]), where $z_{ij}=z_i-z_j$, $\zeta_{ij}=\zeta_i-\zeta_j$,

$$\eta_1 = z_{12}z_{34}\,,\quad \eta_2 = z_{14}z_{23}\,,\quad \eta = \frac{\eta_1}{\eta_1+\eta_2} = \frac{z_{12}z_{34}}{z_{13}z_{24}}\,,$$

$$\xi_1 = \zeta_{12}\zeta_{34}\,,\quad \xi_2 = \zeta_{14}\zeta_{23}\,,\quad \xi = \frac{\xi_1}{\xi_1+\xi_2} = \frac{\zeta_{12}\zeta_{34}}{\zeta_{13}\zeta_{24}}\,\cdot \qquad (2)$$

The function $f_I(\xi,\eta)$ depending only on the harmonic ratios is a *polynomial in* ξ of degree $2I$; we are using the convenient polynomial bases of $SU(2)$ irreducible representations V_I and invariant tensors[1].

The corresponding KZ equation for $f_I(\xi,\eta)$ reads

$$\left((k+2)\eta(1-\eta)\frac{\partial}{\partial\eta} - \sum_{i=0}^{2} K_i^I(\xi,\eta)\frac{\partial^i}{\partial\xi^i}\right) f_I(\xi,\eta) = 0\,,$$

$$K_0^I(\xi,\eta) = 2I\,(2I(1-\xi)-2(I+1)\eta+1)\,,$$

$$K_1^I(\xi,\eta) = (4I-1)\,\xi^2 + 2\,\xi\,(\eta-2I) - \eta\,,$$

$$K_2^I(\xi,\eta) = \xi(1-\xi)(\xi-\eta)\,. \qquad (3)$$

For any I in the range $0 \leq 2I \leq 2k+2$ equation (3) has $2I+1$ (the dimension of $\mathrm{Inv}\, V_I^{\otimes 4}$) linearly independent solutions. We will define the regular basis vectors $w_{I\mu} = w_{I\mu}\,(\underline{\zeta},\underline{z})$, $\mu = 0,\ldots,2I$ as in [3] (including the prefactor $D_I(\underline{\zeta},\underline{z})$) [2].

As mentioned above, the solutions of the KZ equation (3) are given in terms of contour integrals defining, in general, multivalued analytic functions of η, and we are interested in their *monodromy* properties. In fact, the set of $w_{I\mu}$ is closed under *braiding* ("half monodromy") as well. In the case at hand (four equal isospins I) the relevant braid group is $\mathcal{B}_3$ so that there are two

[1] The relation of the latter with the tensor invariants can be illustrated by the correspondence $\xi_1 \leftrightarrow \varepsilon^{A_1A_2}\varepsilon^{A_3A_4}$, $\xi_2 \leftrightarrow \varepsilon^{A_1A_4}\varepsilon^{A_2A_3}$ in the simplest case $I=1/2$ when there are only two independent invariant tensor structures. Here ε^{AB}, $A,B=1,2$ is the two dimensional skew-symmetric tensor spanning $\mathrm{Inv}\, V_{1/2}^{\otimes 2}$. In terms of the harmonic ratio the first invariant corresponds to ξ, and the second – to $1-\xi$.

[2] The conformal block (1) has to be considered as a linear combination of $w_{I\mu}$ with coefficients restricted further by the locality condition imposed on the $2D$ correlation function.

elementary braid generators corresponding, respectively, to the transformations

$$\begin{aligned} B_1 :(\xi_1,\ \xi_2) \ &\to\ (-\,\xi_1,\ \xi_1+\xi_2), \\ (\eta_1,\ \eta_2) \ &\to\ (\mathrm{e}^{-\mathrm{i}\pi}\eta_1,\ \eta_1+\eta_2), \\ B_2 :(\xi_1,\ \xi_2) \ &\to\ (\xi_1+\xi_2,\ -\,\xi_2) \\ (\eta_1,\ \eta_2) \ &\to\ (\eta_1+\eta_2,\ \mathrm{e}^{-\mathrm{i}\pi}\eta_2). \end{aligned} \tag{4}$$

Their action on the regular basis,

$$(B_i^{(k,I)} w)_{I\mu} = (B_i^{(k,I)})^{\lambda}{}_{\mu} w_{I\lambda}\,, \quad i=1,2\,,$$

is given by

$$\begin{aligned} &\left(B_1^{(k,I)}\right)^{\lambda}_{\ \mu} = q^{2I(k+1-I)}(-1)^{\lambda} q^{\lambda(\mu+1)} \begin{bmatrix} \lambda \\ \mu \end{bmatrix}, \\ &\lambda,\ \mu = 0,1,\ldots,2I\,, \quad \begin{bmatrix} \lambda \\ \mu \end{bmatrix} = \frac{[\lambda]!}{[\mu]![\lambda-\mu]!}\,, \\ &[\lambda]! = [\lambda][\lambda-1]! \quad (\,[0]!=1\,)\,, \quad [\lambda] = \frac{q^{\lambda}-q^{-\lambda}}{q-q^{-1}} \end{aligned} \tag{5}$$

(we will fix $q = \mathrm{e}^{-\mathrm{i}\frac{\pi}{k+2}}$, see [26]) and

$$B_2^{(k,I)} = F^I\, B_1^{(k,I)}\, F^I\,,\ (F^I)^{\lambda}{}_{\mu} = \delta^{\lambda}_{2I-\mu} \quad ((F^I)^2 = \mathbb{1})\,. \tag{6}$$

The matrices $B_1^{(k,I)}$, $B_2^{(k,I)}$ are lower, respectively, upper triangular.

The rationality condition implies that $f_I(\xi,\eta)$ is a polynomial in η of order not exceeding $4\Delta_I$ [1,2]. If Δ_I (and hence, I) is (half-)integer, polynomial solutions of the KZ equation (3) give rise to $\mathcal{B}_3$ invariants (up to a sign, for half-integer Δ_I). Hence (see (1)),

$$(-1)^{2I} f_I(1-\xi,1-\eta) = f_I(\xi,\eta) = \xi^{2I}(-\eta)^{4\Delta_I} f_I(\frac{1}{\xi},\frac{1}{\eta})\,. \tag{7}$$

3 Braid invariant functions in terms of the regular basis

All polynomial solutions of the KZ equation (3) for $0 \le 2I \le k$ (satisfying the initial condition $f_I(\xi,0)=\xi^{2I}$ following from the factorization of the four-point function into a product of two-point functions for $\eta\to 0$) have been found in [1,2]. The list includes the simple currents series existing for any k

$$f_{k/2}(\xi,\eta) = (\xi-\eta)^k \quad (I=k/2\,,\ \Delta_{k/2}=k/4) \tag{8}$$

which, for integer $\Delta_{k/2}$, corresponds to the D_{even} series in the ADE classification, and a few exceptional cases occuring for $k=10$, $I=2,3$ and $k=28$, $I=5,9$ (corresponding to E_6 and E_8 , respectively; see [1] for explicit expressions). Except for the solutions in (8) at odd values of k , all the rest give rise to rational functions. How could the latter be expressed in terms of the multivalued functions of the regular basis? To answer this question, one can make use of their simple braiding properties. Taking into account the prefactors as well, for all KZ solutions (8) (including those for odd k) one has

$$\begin{aligned} &s^{(k,k/2)} = s^{(k,k/2)}(\underline{\zeta},\underline{z}) = D_{k/2}(\underline{\zeta},\underline{z})\, f_{k/2}(\xi,\eta)\,, \\ &s^{(k,k/2)} = s^{\mu} w_{k/2\ \mu}\,, \quad (B_{1,2}^{(k,k/2)})^{\lambda}{}_{\mu} s^{\mu} = (-\mathrm{i})^k s^{\lambda}\,, \end{aligned} \tag{9}$$

i.e., $s^{(k,k/2)}$ are common eigenvectors of $B_i^{(k,k/2)}$, $i=1,2$ (see Eqs. (5, 6)) corresponding to the eigenvalue $(B_1^{(k,k/2)})^0_0 = (B_2^{(k,k/2)})^k_k$ and hence the coefficients s^{μ} of their expansion in terms of the regular basis can be found, up to an overall coefficient, by solving a finite dimensional eigenvector problem. This can be easily done, and the solution is (proportional to)

$$s^{\mu} = \frac{(-1)^{\mu}}{[\mu+1]}\,, \quad \mu=0,\ldots,k \tag{10}$$

– one has to make use of a well known q-binomial identity written in the form

$$\sum_{\mu=0}^{k} (-1)^{\mu} q^{\lambda(\mu+1)} \begin{bmatrix} \lambda+1 \\ \mu+1 \end{bmatrix} = 1 \quad \text{for } 0 \le \lambda \le k\,. \tag{11}$$

For the (E_6) case $(k,I)=(10,3)$ one gets $s^0=s^6=1$, $s^1=s^5=-\frac{1}{[2]}$, $s^2=s^4=\frac{1}{[3]}$, $s^3=\frac{3}{[3]}\,\frac{-1}{[4]}$. (The apparent symmetry of the coefficients which is a general property can be easily understood since (s^{μ}) should be an eigenvector of the antidiagonal matrix F (6) as well [28].)

This result can be made more explicit in the (nonrational) case $k=1=2I$ for which the elements of the regular basis are known in terms of hypergeometric functions [26] where it leads to the identity

$$\begin{aligned} &(1-\eta)^{\frac{2}{3}}\,(\,2\,\xi\, F_1(1-\eta) \,+\, (1-\xi)\,\eta\, F_2(1-\eta)\,) \\ &-\eta^{\frac{2}{3}}\,(\,2\,(1-\xi)\, F_1(\eta) \,+\, \xi\,(1-\eta)\, F_2(\eta)\,) \\ &= \frac{2}{3}\, B\left(\frac{2}{3},\frac{2}{3}\right)\,(\xi-\eta)\,. \end{aligned} \tag{12}$$

Here $F_1(\eta) = F\left(\frac{1}{3},-\frac{1}{3};\frac{2}{3};\eta\right)$, $F_2(\eta) = F\left(\frac{4}{3},\frac{2}{3};\frac{5}{3};\eta\right)$ and $B(x,y)$ is the beta function.

4 Polynomial solutions of the KZ equation for I = k + 1

We have found polynomial solutions of the KZ equation (3) for $I=k+1=\Delta_{k+1}$ as well – a value of I corresponding to a non-integrable representation of the current algebra, the counterpart of the vacuum representation under the duality $I \leftrightarrow k+1-I$. These solutions give rise to braid invariant vectors (s^μ), $\mu = 0, \ldots, 2(k+1)$ whose only nonzero coefficient is s^{k+1} and hence, in contrast with the cases considered in the previous section, the corresponding rational functions belong to the regular basis. This fact might be important since it can shed some light on the corresponding logarithmic CFT as well.

It is easy to obtain directly the polynomial solutions of the KZ equation (3) corresponding to $I = k+1$ for lower values of k. Extrapolating the results, one arrives to the expression (throughout this paragraph $I \equiv k+1$)

$$\begin{aligned} f_I(\xi,\eta) &= (\eta(1-\eta))^I \sum_{m=0}^{2I}\sum_{n=0}^{I} C^I_{mn}\xi^m\eta^n \\ &\equiv (\eta(1-\eta))^I\, p_I(\xi,\eta) \end{aligned} \tag{13}$$

where $p_I(\xi,\eta)$ are polynomials of order $2I$ in ξ and of order I in η, the coefficients C^I_{mn} being chosen as

$$C^I_{mn} = (-1)^{I+m+n}\binom{I}{m+n-I}\binom{m+n}{n}\binom{3I-m-n}{I-n}. \tag{14}$$

Note that the overall order of $f_I(\xi,\eta)$ in η is $3I$ *i.e.*, strongly below the upper bound $4\Delta_I$. One can check directly that the polynomial (13) solves the KZ equation (3) for $I = k+1$. The expression for $p_I(\xi,\eta)$ following from (13, 14) can be brought to the form

$$p_I(\xi,\eta) = \sum_{n=0}^{I}\binom{I}{n}\eta^{I-n}\xi^n P_{In}(\xi) \tag{15}$$

where

$$\begin{aligned} P_{In}(\xi) &= \sum_{\ell=0}^{I} a_{n\ell}\,\xi^\ell, \\ a_{n\ell} &= (-1)^\ell \frac{(I+\ell)!\,(2I-\ell)!}{\ell!\,(I-\ell)!(n+\ell)!(2I-n-\ell)!}. \end{aligned} \tag{16}$$

Indeed,

$$\begin{aligned} p_I(\xi,\eta) &= \sum_{n=0}^{I}\eta^{I-n}\xi^n\sum_{m=n}^{2I}(-1)^{m-n} \\ &\times \binom{I}{m-n}\binom{I+m-n}{m}\binom{2I-m+n}{2I-m}\xi^{m-n} \\ &= \sum_{n=0}^{I}\eta^{I-n}\xi^n\sum_{\ell=0}^{2I-n}(-1)^\ell\binom{I}{\ell}\binom{I+\ell}{n+\ell}\binom{2I-\ell}{2I-n-\ell}\xi^\ell \end{aligned} \tag{17}$$

which is equivalent to (16). The form of the coefficients $a_{n\ell}$ (16) implies that

$$P_{In}(\xi) = \frac{(-1)^n}{n!}\xi^{2I+1}\frac{\mathrm{d}^n}{\mathrm{d}\xi^n}(\xi^{n-2I-1}F(-I,I+1;n+1;\xi)). \tag{18}$$

At these special integer values of the parameters the hypergeometric functions in (18) are expressible in terms of Jacobi polynomials $\mathcal{P}^{(\alpha,\beta)}_\ell(x)$ [30],

$$\begin{aligned} &\binom{n+I}{I}F(-I,I+1;n+1;\xi) = \mathcal{P}^{(n,-n)}_I(1-2\xi) \\ &\equiv \sum_{\ell=0}^{I}(-1)^{I-\ell}\binom{I+n}{\ell}\binom{I-n}{I-\ell}\xi^{I-\ell}(1-\xi)^\ell \\ &= \frac{1}{I!}\left(\frac{1-\xi}{\xi}\right)^n\frac{\mathrm{d}^I}{\mathrm{d}\xi^I}\left(\xi^{I+n}(1-\xi)^{I-n}\right). \end{aligned} \tag{19}$$

It can be checked that $f_I(\xi,\eta)$ satisfies the two relations of equation (7). To prove this, one can use the explicit expression (14) for the coefficients $C^I_{mn} = (-1)^I C^I_{2I-m\ I-n}$ and some combinatorial identities. Checking the second relation, one should also have in mind that $f_I(\xi,\eta)$ only contains powers of η in the range between I and $3I$.

More details concerning the role of the corresponding rational functions as vectors from the regular basis will be given elsewhere [29].

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